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A PERTURBATION SOLUTION OF THE MAIN PROBLEM
IN ARTIFICIAL SATELLITE THEORY

by

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A Perturbation Solution of the Main Problem in Artificial Satellite Theory

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The main problem of artificial satellite theory is a restricted two body problem in which the Legendre Polynomial representation of the cylindrically symmetric potential contains only the first two terms. A generalized asymptotic expansion is used to obtain a first order approximation. The solution at the critical inclination is seen to be of a different type than at other inclinations. The solution is finite for all eccentricities and inclinations when suitably restricted in time.

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NOTATION

α	right ascension in spherical coordinates
β	declination in spherical coordinates
γ	vernal equinox
c	$\cos i_0$
ϵ_0	eccentricity of the initial instantaneous ellipse
h_0	initial magnitude of angular momentum
i	inclination
i_0	initial inclination
J_2	coefficient of second Legendre polynomial in the representation of the gravitational potential
J	$3/2(J_2R^2/P_0^2)$
K_i	undetermined constants used in the solution
p_0	h_0^2/GM where G is the gravitational constant and M is the mass of the primary body
R	equatorial radius of the primary body
r	magnitude of the position vector to the satellite
s	$\sin i_0$
T	kinetic energy of the satellite
t	time
t_0	initial value of time
u	p_0/r
V	gravitational potential
ω_0	initial value of the argument of perigee
θ	angle measured from the ascending node to the satellite within the reference plane
θ_0	initial value of θ
Ω	longitude of the ascending node measured from γ
Ω_0	initial value of Ω

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Blaise Pascal, the famous French mathematician and philosopher, wrote that when people refer to *my* house or *my* book, etc., that it would be better if they were to use *our* rather than *my* since it usually happens that we ourselves contribute but a small part to an effort. Such is the case here.

Most special thanks to my mentor and friend Professor Don Danielson, who guided me through the calculations with ease and confidence. Thanks also to LTC James R. Snider, whose work on this problem revealed the various pitfalls in the method. It is nice to travel a road that has been traveled before.

Special thanks to my wife Gail Weiss for her support throughout this endeavor.

A. INTRODUCTION

The main problem of artificial satellite theory is a restricted two body problem in which the Legendre Polynomial representation of the cylindrically symmetric potential contains only the first two terms. Deceptively simple in statement, the main problem continues to evade satisfactory solution. We make no attempt to survey the vast literature pertaining to the main problem, nor to recount the history of the so-called critical inclination. In fact, every effort is made to make the solution as accessible to the non-specialist as to the astrodynamicist.

Our goals in attempting to obtain a solution are twofold. First, we desire a solution employing a coordinate system based upon physical events rather than an abstract set of transformations. For instance, the use of averaged quantities such as the "mean orbital plane" or other such non-physical artifices is eschewed. Second, we seek a solution which does not tend to infinity at any inclination and which places no constraints on any dependent variable.

Needed for the solution are a coordinate system, asssumptions concerning the dependent variables involved, and an algorithm for calculating the unknowns. These are outlined in detail in the first chapters, after which the solution is obtained through extensive use of MACSYMA. A discussion of the critical inclination problem follows.

Because the purpose and scope of this work are nearly identical to those of Snider [Ref. 1], there are many similarities in organization, notation, and language. The principal difference in treatment lies in the geometry used rather than in approach. Snider's original approach is succesfully employed using an alternate geometry, and his influence in this work is pervasive.

B. ORBITAL KINEMATICS

Figure 1 shows the reference system of spherical coordinates (r, α, β). The radial distance r is measured from the center of the planet O to the satellite S. The line $O\gamma$ is in a direction fixed with respect to an inertial coordinate system. The right ascension α is the angle measured in the planet's equatorial plane eastward from the line $O\gamma$. The declination or latitude β is the angle measured northward from the equator. The position vector $\mathbf{r}$ of the satellite in the spherical coordinate system is

$$\mathbf{r} = r(\cos\alpha\cos\beta)\mathbf{b}_1 + r(\sin\alpha\cos\beta)\mathbf{b}_2 + r(\sin\beta)\mathbf{b}_3 \tag{1}$$

where $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ are orthonormal base vectors fixed in the directions shown.

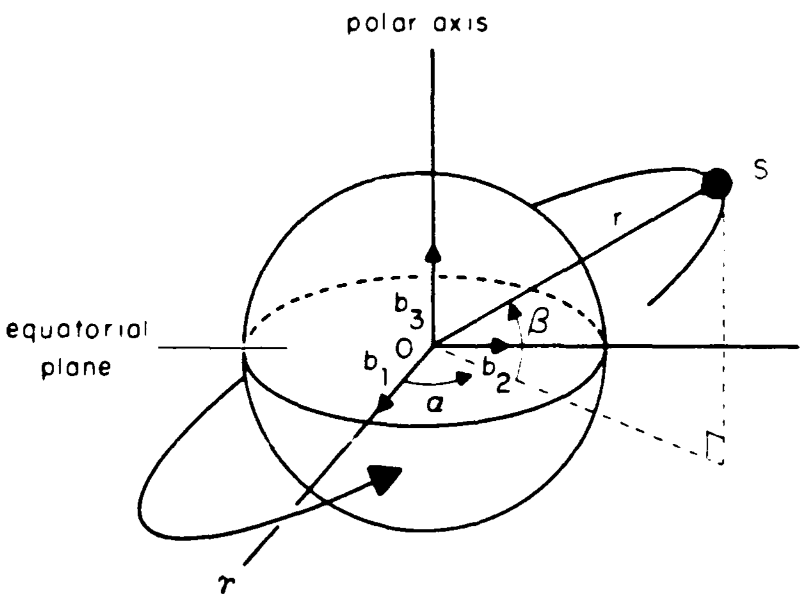

Figure 1: Spherical coordinate system.

Figure 2 shows how we can locate the satellite by its polar coordinates (r, θ) within a rotating orbital plane that contains its position and velocity vectors. Here θ is the argument of latitude, i.e., the angle measured in the orbital plane from the ascending node to the satellite. The orbital plane is inclined at an angle i to the

equatorial plane and intersects the equatorial plane in the line of nodes, making an angle Ω with the $O\gamma$ line.

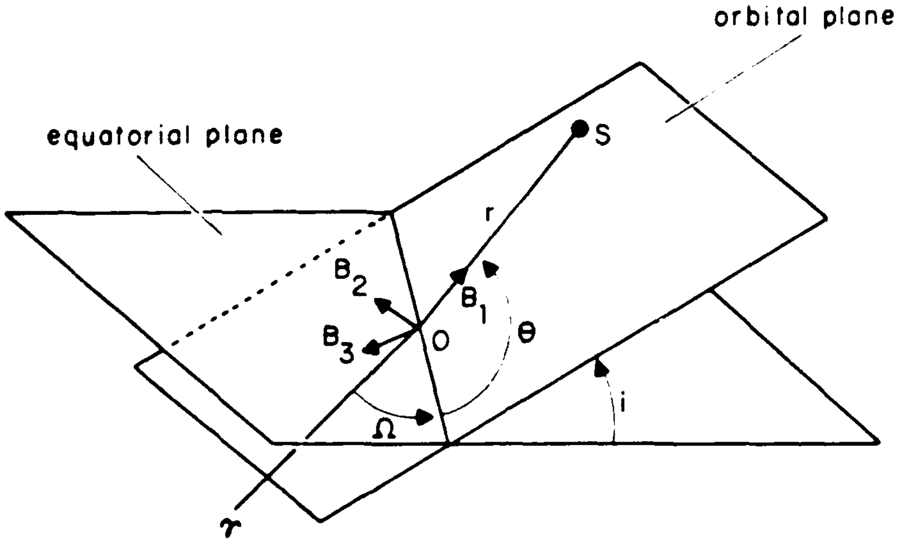

Figure 2: Orbital plane

We introduce another orthonormal set of basis vectors $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ which move with the satellite so that $\mathbf{B}_1$ is in the direction of the position vector $\mathbf{r}$, $\mathbf{B}_2$ is also in the orbital plane and $\mathbf{B}_3 = \mathbf{B}_1 \times \mathbf{B}_2$.

The basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ may be transformed into the basis $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ by a succession of three rotations. First the basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is rotated about the $\mathbf{b}_3$ direction by the angle Ω, next the basis is rotated about the new first coordinate vector by the angle i, and finally the basis is again rotated about the new third coordinate vector by the angle θ. The two sets of base vectors are related by the product of the rotation matrices representing each successive rotation:

$$\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} \quad (2)$$

or

$$\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\Omega - \sin\theta\cos i\sin\Omega & \cos\theta\sin\Omega + \sin\theta\cos i\cos\Omega & \sin\theta\sin i \\ -\sin\theta\cos\Omega - \cos\theta\cos i\sin\Omega & -\sin\theta\sin\Omega + \cos\theta\cos i\cos\Omega & \cos\theta\sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}$$

The position vector $\mathbf{r}$ has only one component in the rotating basis:

$$\mathbf{r} = r\mathbf{B}_1 \tag{3}$$

Using the first of Equations (2), we obtain the components of $\mathbf{r}$ in the fixed basis:

$$\mathbf{r} = r(\cos\theta\cos\Omega - \sin\theta\cos i\sin\Omega)\mathbf{b}_1$$

$$+ r(\cos\theta\sin\Omega + \sin\theta\cos i\cos\Omega)\mathbf{b}_2 + r(\sin\theta\sin i)\mathbf{b}_3 \tag{4}$$

Equating the components of Equations (1) and (4), we can obtain the following relations among the angles (α, β) of the spherical coordinate system and the astronomical angles (i, Ω, θ):

$$\sin\beta = \sin\theta\sin i \tag{5}$$

$$\cos\beta = \cos\theta\sec(\alpha - \Omega)$$

The velocity $d\mathbf{r}/dt$ of the satellite is obtained by differentiating (3) with respect to the time t:

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{B}_1 + r\frac{d\mathbf{B}_1}{dt} \tag{6}$$

Since the orbital plane must contain the velocity vector, we have to enforce

$$\frac{d\mathbf{B}_1}{dt} \cdot \mathbf{B}_3 = 0 \tag{7}$$

Substitution of Equation (2) into Equation (7) leads to a relationship which uncouples the equations for $\Omega(\theta)$ and $i(\theta)$:

$$\frac{d\Omega}{d\theta} = \frac{\tan\theta}{\sin i}\,\frac{di}{d\theta} \tag{8}$$

The velocity (6) can then be written

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{B}_1 + r\frac{d\theta}{dt}\left(1 + \tan\theta \cot i \frac{di}{d\theta}\right)\mathbf{B}_2 \tag{9}$$

In the following part of this paper, we obtain expressions for $r(\theta)$, $i(\theta)$, $\Omega(\theta)$, and $dt/d\theta(\theta)$. The position and velocity vectors of the satellite are then calculated from the formulas in this chapter. The classical orbital elements p, e, and ω are the semilatus rectum, eccentricity, and argument of perigee of the instantaneous (osculating) ellipse determined by the position and velocity vectors. If needed, $p(\theta)$, $e(\theta)$, and $\omega(\theta)$ can be obtained from our solution $r(\theta)$ and $dt/d\theta(\theta)$:

$$p = \frac{r^4}{GM\left(\frac{dt}{d\theta}\right)^2}$$

$$e\cos(\theta - \omega) = \frac{p}{r} - 1$$

$$e\sin(\theta - \omega) = \frac{p}{r^2}\left(\frac{dr}{d\theta}\right)$$

Numerical integration of the expression for $dt/d\theta(\theta)$ allows a transformation between θ and t. Figure 3 shows the satellite at time t and its relationship to the various variables used.

This method employs the *true*, rather than *mean*, orbital plane to specify the satellite's position and is due to Struble [Ref. 2, 3, 4].

C. EQUATIONS OF MOTION

The expressions in spherical coordinates for the kinetic and potential energies per unit mass of a satellite orbiting around an oblate planet are respectively:

$$T = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\beta}{dt}\right)^2 + r^2\cos^2\beta\left(\frac{d\alpha}{dt}\right)^2\right] \tag{10}$$

$$V = -\frac{GM}{r}\left[1 + \frac{J_2 R^2}{2r^2}\left(1 - 3\sin^2\beta\right)\right] \tag{11}$$

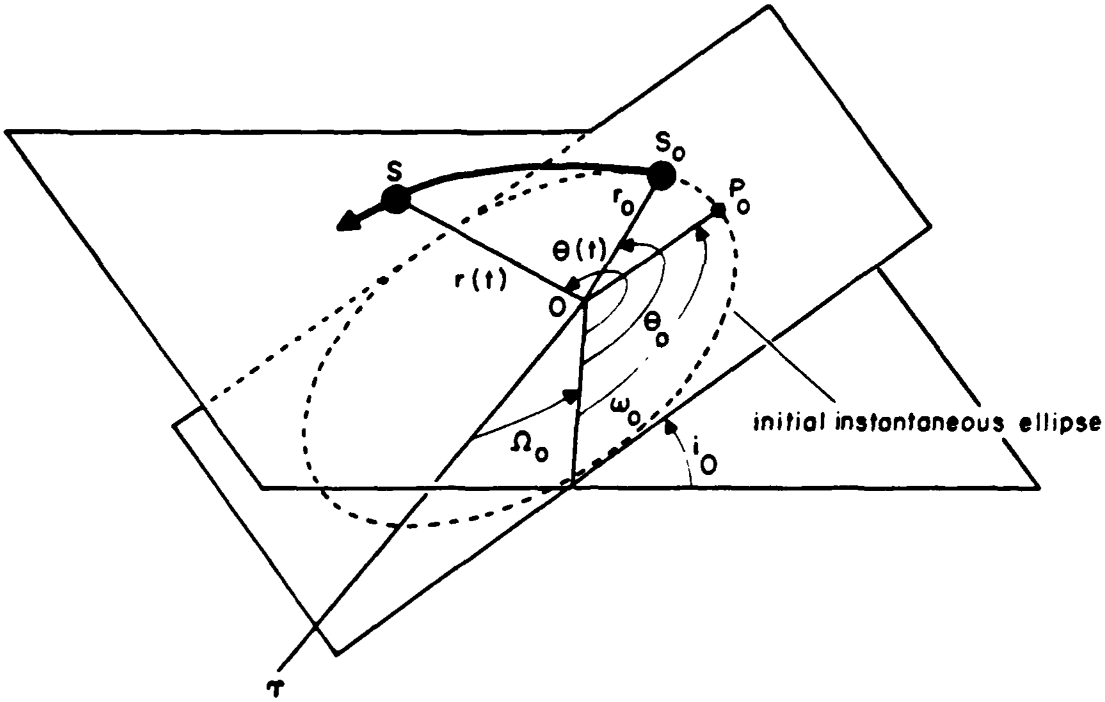

Figure 3: Satellite at time t.

where G is the gravitational constant, M is the mass of the planet, and R is the equatorial radius of the planet. Substitution of these equations into Lagrange's equations

$$\frac{d}{dt}\frac{\partial(T-V)}{\partial\left(\frac{dq}{dt}\right)} - \frac{\partial}{\partial q}(T-V) = 0 \qquad q = r, \alpha, \text{ or } \beta$$

results in the following equations of motion:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\beta}{dt}\right)^2 - r\cos^2\beta\left(\frac{d\alpha}{dt}\right)^2 = -\frac{\partial V}{\partial r} \tag{12}$$

$$\frac{d}{dt}\left(r^2\cos^2\beta\frac{d\alpha}{dt}\right) = 0$$

$$\frac{d}{dt}\left(r^2\frac{d\beta}{dt}\right) + r^2\sin\beta\cos\beta\left(\frac{d\alpha}{dt}\right)^2 = -\frac{\partial V}{\partial\beta} \tag{13}$$

Initial conditions are established by requiring that at the initial time t_0 the orbital parameters of the Keplerian two body ellipse, determined by the initial position and velocity vectors, are known. The actual orbit is then tangent to this initial instantaneous ellipse (see Figure 3). Equating the initial position and velocity vectors given by Equations (3) and (9) to the two body expressions, we obtain

$$r(t_0) = \frac{p_0}{1 + e_0 \cos(\theta_0 - \omega_0)}, \qquad \frac{dr}{dt}(t_0) = \frac{e_0 h_0 \sin(\theta_0 - \omega_0)}{p_0} \tag{14}$$

$$\frac{d\theta}{dt}(t_0) = \frac{h_0}{r_0^2 \left[1 + \tan\theta_0 \cot i_0 \frac{di}{d\theta}(\theta_0)\right]} \tag{15}$$

$$i(\theta_0) = i_0 \tag{16}$$

$$\Omega(\theta_0) = \Omega_0 \tag{17}$$

Here $h_0 = \sqrt{GMp_0}$ is the initial value of the satellite's *specific angular momentum* about the center of the planet. The subscript 0 on a symbol denotes that the parameter is evaluated at the initial time t_0.

We immediately have two integrals of the equations of motion:

$$T + V = \text{constant} \tag{18}$$

and

$$r^2 \cos^2 \beta \frac{d\alpha}{dt} = \text{constant} \tag{19}$$

Equation (18) states that the mechanical energy of the satellite remains constant. Now, from Equations (1) and (19)

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} \cdot \mathbf{b}_3 = r^2 \cos^2 \beta \frac{d\alpha}{dt} = h_0 \cos i_0 \tag{20}$$

Equation (20) states that component along the polar axis of the specific angular momentum of the satellite remains constant. Inserting Equations (3) and (9) into Equation (20), and applying the initial condition (15), we obtain

$$\frac{dt}{d\theta} = \frac{r^2 \cos i}{h_0 \cos i_0} \left(1 + \tan\theta \cot i \frac{di}{d\theta}\right) \tag{21}$$

This allows the independent variable to be changed from t to θ.

Letting $u = p_0/r$, and using Equations (5), (20), and (21), we can rewrite the remaining equations of motion (12)-(13):

$$\frac{di}{d\theta} = \frac{-2J u \sin\theta \cos\theta \sin i \cos^2 i}{\frac{c^2}{\cos i} + 2Ju \sin^2\theta \cos^3 i} \tag{22}$$

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u = \frac{\cos^2 i}{c^2} + \frac{J\cos^2 i}{c^2}\Big[u^2(1 - 3\sin^2\theta \sin^2 i) \\ + 2u\frac{du}{d\theta}\sin\theta\cos\theta(1 - 3\cos^2 i) - 4u\frac{d^2u}{d\theta^2}\sin^2\theta\cos^2 i - 2\left(\frac{du}{d\theta}\right)^2 \sin^2\theta\cos^2 i\Big] \\ - \frac{4J^2 u \sin^3\theta \cos^6 i}{c^4}\left[u\frac{du}{d\theta}\cos\theta(2 + \sin^2 i) + \frac{d}{d\theta}\left(u\frac{du}{d\theta}\right)\sin\theta\cos^2 i\right] \end{aligned} \tag{23}$$

The terms with $d^2u/d\theta^2$ on the right side of (23) can be eliminated, yielding the equivalent equation

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u = \Big\{c^2\cos^2 i - Jc^2 u \cos^2 i\left[2\frac{du}{d\theta}\sin\theta\cos\theta(3\cos^2 i - 1) - u\right] \\ - 4J^2u^2\frac{du}{d\theta}\sin^3\theta\cos\theta\cos^6 i(3 - \cos^2 i)\Big\} \\ /(c^4 + 4Juc^2\sin^2\theta\cos^4 i + 4u^2J^2\sin^4\theta\cos^8 i) \end{aligned} \tag{24}$$

Here we have introduced the shorthand notation $c = \cos i_0$, $s = \sin i_0$, $J = 3J_2R^2/2p_0^2$.

The differential Equations (22)-(23) are coupled by the nonlinear terms and apparently cannot be solved analytically. If we expand the right sides of (22) and (24) in a Taylor series expansion in powers of J, the equations simplify to

$$\frac{di}{d\theta} = \frac{-2Ju\sin\theta\cos\theta\sin i\cos^3 i}{c^2} + 4J^2u^2sc^3\sin^3\theta\cos\theta + O(J^3) \tag{25}$$

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u = \frac{\cos^2 i}{c^2} + \frac{J\cos^2 i}{c^2}\Bigg\{&\frac{-4u\sin^2\theta\cos^4 i}{c^2} + u^2[1+\sin^2\theta(7\cos^2 i - 3)] \\ &+ 2u\frac{du}{d\theta}\sin\theta\cos\theta(1-3\cos^2 i) - 2\left(\frac{du}{d\theta}\right)^2\sin^2\theta\cos^2 i\Bigg\} \\ + \frac{4J^2u\sin^2\theta\cos^6 i}{c^4}&\Bigg\{u^2[-1+3\sin^2\theta(1-2\cos^2 i)] + \frac{3u\sin^2\theta\cos^4 i}{c^2} \\ + u\frac{du}{d\theta}\sin\theta\cos\theta[4\sin^2 i &+ \sin^2\theta(1-3\cos^2 i)] + \left(\frac{du}{d\theta}\right)^2\sin^2\theta\cos^2 i\Bigg\} + O(J^3)\end{aligned} \tag{26}$$

Using (8) together with (25) yields

$$\frac{d\Omega}{d\theta} = -2Juc\sin^2\theta + 4J^2u^2c^3\sin^4\theta + O(J^3) \tag{27}$$

Each of the neglected terms in Equations (25)-(27) are indicated by the O symbols. These are terms which will be multiplied by J to the third power or higher. Note that the Equations (25)-(26) are identical to those used as the starting point in the analysis of Eckstein, Shi, and Kevorkian [Ref. 5].

D. METHOD OF SOLUTION

We will use a perturbation technique to solve Equations (25)-(27). Following Erdélyi [Ref. 6], we define the *order relations* O and o as follows. For two real-valued functions $f(x)$ and $g(x)$, we define the relation $f = O(g)$ if there exists some real constant ξ such that $|f| \le \xi|g|$ for all x in some domain of interest. Similarly, we define $f = o(g)$ as $x \to x_o$ if for all $\epsilon > 0$, there exists some neighborhood of x_o such that $|f| \le \epsilon|g|$ within that neighborhood.

The series $\sum_n a_n(\theta, J)J^n$ is said to be a generalized asymptotic approximation to N terms of $f(\theta, J)$ with respect to the scale $\{J^n\}$ as $J \to 0$ if

$$f(\theta, J) = \sum_{n=0}^{N} a_n(\theta, J)J^N + o(J^N) \text{ as } J \to 0$$

Let D be some subset of the real line to be determined. We will say that the above generalized asymptotic approximation holds uniformly for $\theta \in$ D if

$$f(\theta, J) - \sum_{n=0}^{N} a_n(\theta, J)J^N = o(J^N) \text{ as } J \to 0$$

uniformly for $\theta \in$ D. In order to get uniformity it will sometimes be necessary to bound the independent variable θ, thus determining D. Note that these definitions differ from the usual asymptotic expansion definitions in that we allow the coefficient functions a_i to be functions of both θ and J.

Let us suppose that each of i, u, and Ω have formal generalized asymptotic expansions for some suitably restricted values of θ:

$$i \approx \sum_{k=0}^{\infty} i_k(\theta, J)J^k \tag{28}$$

$$u \approx \sum_{k=0}^{\infty} u_k(\theta, J)J^k \tag{29}$$

$$\Omega \approx \sum_{k=0}^{\infty} \Omega_k(\theta, J)J^k \tag{30}$$

Brenner and Latta [Ref. 7] found that a modification was needed to their time-like variable M in order to avoid resonance. They accomplished this by means of an additional variable ω which multiplied M, allowing secular term elimination. Analogously we introduce the variable y which may be added to integer multiples of θ. In this way we too will avoid resonances except perhaps at certain inclinations.

The explicit form is then

$$y \approx \sum_{k=0}^{\infty} y_k(\theta, J) J^k \tag{31}$$

We further stipulate that for $J = 0$, u assumes the Keplerian two body solution and that y is the true anomaly. This forces

$$y \approx \underbrace{\theta - \omega_o}_{\text{true anomaly}} + Jy_1 + J^2 y_2 + \cdots \tag{32}$$

$$u \approx \underbrace{1 + e_0 \cos y}_{\text{Keplerian solution}} + Ju_1 + J^2 u_2 + \cdots \tag{33}$$

An algorithm for the perturbation procedure is then:

Let n=1
Substitute Equations (28), (30), (32), (33) into the equations of motion
Equate coefficients of J^n
Solve for the n^{th} *order solution*
Iterate on n

Having chosen a coordinate system, made assumptions concerning the variables, and given an algorithm for determining the unknowns, we are prepared to give the solution in the following chapter.

E. SOLUTION

Substituting Equations (33) and (28) into (25) and equating the terms multiplied by J yields

$$\frac{di_1}{d\theta} = -sc \sin 2\theta - \frac{sce_0}{2} \sin(y + 2\theta) + \frac{sce_0}{2} \sin(y - 2\theta) \tag{34}$$

A solution to this equation is

$$i_1 = \frac{sc}{2} \cos 2\theta + \frac{sce_0}{6} \cos(y + 2\theta) + \frac{sce_0}{2} \cos(y - 2\theta)$$

$$+ K_1 \cos(2y - 2\theta) + K_2 \tag{35}$$

The last two terms may be added because they are, to lowest order, homogenous solutions to Equation (34). The term multiplied by the constant K_1 was added to eliminate secular terms in i_2. Note that differentiating this term with respect to θ produces terms multiplied by J, from Equation (32). The constant K_2 was added to satisfy the initial condition (16), which implies that $i_1(\theta_0) = 0$ so

$$K_2 = -\frac{sc}{2}\cos 2\theta_0 - \frac{sce_0}{6}\cos(3\theta_0 - \omega_0) - \frac{sce_0}{2}\cos(\theta_0 + \omega_0) - K_1 \cos 2\omega_0$$

Substituting Equations (28), (32), (33), and (35) into (26), and equating terms multiplied by J yields

$$\begin{aligned}\frac{d^2u_1}{d\theta^2} + u_1 = 1 - \frac{3s^2}{2} + e_0^2\left(-\frac{5s^2}{4} + 1\right) + \frac{1}{4}[(2 + 5e_0^2)s^2 - 2e_0^2]\cos 2\theta \\ + \frac{e_0^2}{4}(-9s^2 + 8)\cos 2y + \frac{e_0}{3}(11s^2 - 6)\cos(y + 2\theta) \qquad (36) \\ + \frac{15e_0^2}{24}(3s^2 - 2)\cos(2y + 2\theta) + \left[\frac{e_0^2}{8}(3s^2 - 2) - \frac{2sK_1}{c}\right]\cos(2y - 2\theta) \\ - \frac{2sK_2}{c} + e_0\left(2\frac{dy_1}{d\theta} + 4 - 5s^2\right)\cos y + e_0\frac{d^2y_1}{d\theta^2}\sin y\end{aligned}$$

In the above equation, the $\cos y$ and $\sin y$ terms would produce secular terms $\theta \sin y$ and $\theta \cos y$ in u_1. Choosing $dy_1/d\theta = 5s^2/2 - 2$ will eliminate these possibilities. Integrating yields

$$y_1 = \left(\frac{5s^2}{2} - 2\right)(\theta - \theta_0) + K_3[\sin(2y - 2\theta) + \sin 2\omega_0] \qquad (37)$$

The term multiplied by K_3 was added to eliminate secular terms in u_2. The constant terms in (37) were added to satisfy the initial condition $y(\theta_0) = \theta_0 - \omega_0$.

A solution to lowest order of Equation (36) is then

$$\begin{aligned}u_1 = 1 - \frac{3s^2}{2} + e_0^2\left(\frac{-5s^2}{4} + 1\right) + \frac{1}{12}[-s^2(2 + 5e_0^2) + 2e_0^2]\cos 2\theta \\ + \frac{e_0^2}{12}(9s^2 - 8)\cos 2y + \frac{e_0}{24}(-11s^2 + 6)\cos(y + 2\theta)\end{aligned}$$

$$+\frac{e_0^2}{24}(-3s^2+2)\cos(2y+2\theta)+\left[\frac{e_0^2}{8}(3s^2-2)-\frac{2sK_1}{c}\right]\cos(2y-2\theta) \tag{38}$$

$$-\frac{2sK_2}{c}+K_4\cos(y-2\theta)$$

$$+K_5\cos(y-\theta_0+\omega_0)+K_6\sin(y-\theta_0+\omega_0)$$

The term multiplied by K_4 was added to eliminate secular terms in u_2. The terms multiplied by K_5 and K_6 were added to satisfy the initial conditions (14).

The calculations proceed by substituting Equations (28), (32), (33), (35), (37), and (38) into (25) and equating terms multiplied by J^2:

$$\frac{di_2}{d\theta}=\left[K_1+\frac{sce_0^2(15s^2-14)}{24(5s^2-4)}\right]\sin(2y-2\theta)+\dots \tag{39}$$

For brevity we have indicated on the right side of Equation (39) only the term that would produce secular terms in i_2. Removal of this term by making its coefficient zero determines K_1. Equation (39) is then integrated to determine i_2.

It should be noted that i_2 is needed for the determination of the constants K_3 and K_4 in the equation for u_2, so integration is required.

Continuing the procedure by equating the terms multiplied by J^2 in the expansion of Equation (26) determines y_2, K_3, and K_4. Final values of all the constants are listed in the Appendix.

$\Omega(\theta)$ is determined by substituting (30) into (27) and proceeding in the same way as above. Note that terms in $J^2\theta$ arise in $\Omega(\theta)$. These must be retained and will restrict our variable θ accordingly.

In the form below, use has been made of trigonometric identities in order to isolate terms containing the quantity $(5s^2-4)$ in the denominator. It may be seen that when $(5s^2-4)$ is zero, each of the variables has a finite limit. Note the necessity of keeping the $J^2\theta$ terms in y and Ω.

When the quantity $(5s^2-4)$ is zero, the terms multiplied by J and $J^2\theta$ combine, allowing the division by $(5s^2-4)$ to occur.

$$
\begin{aligned}
r \;=\; & p_0/\Bigg\{1 + e_0\cos y + J\Bigg[1 - \frac{3s^2}{2} + e_0^2\left(1 - \frac{5s^2}{4}\right) \\
+\; & \frac{1}{12}(-(2+5e_0^2)s^2 + 2e_0^2)\cos 2\theta + \frac{e_0^2}{12}(9s^2-8)\cos 2y \\
+\; & \frac{e_0}{24}(-11s^2+6)\cos(y+2\theta) + \frac{e_0^2}{24}(-3s^2+2)\cos(2y+2\theta) \\
+\; & \frac{e_0^2}{8}(3s^2-2)\cos(2y-2\theta) \\
+\; & \frac{e_0[15(2+e_0^2)s^4 - 14(4+e_0^2)s^2 + 24]\sin\left[J\theta\left(\frac{5s^2}{2}-2\right)\right]\sin[\theta+\omega_0]}{12(5s^2-4)} \\
+\; & \frac{e_0^2 s^2(15s^2-14)\sin\left[J\theta\left(\frac{5s^2}{2}-2\right)\right]\sin\left[2\omega_0 - J\theta\left(\frac{5s^2}{2}-2\right)\right]}{6(5s^2-4)} \\
-\; & \frac{e_0^2 s^2}{16}\cos(y-\theta_0+3\omega_0) + \frac{e_0^2}{24}(3s^2-2)\cos(y-3\theta_0+3\omega_0) \\
-\; & \frac{e_0^2 s^2}{16}\cos(y-5\theta_0+3\omega_0) + \frac{e_0}{4}(3s^2-2)\cos(y-2\theta_0+2\omega_0) \\
-\; & \frac{3e_0 s^2}{8}\cos(y-4\theta_0+2\omega_0) - \frac{e_0}{4}(s^2+1)\cos(y+2\omega_0) \qquad (40) \\
+\; & \frac{1}{8}[(-2+5e_0^2)s^2 - 2e_0^2]\cos(y+\theta_0+\omega_0) \\
+\; & \frac{1}{4}[6+5e_0^2)s^2 - 4(1+e_0^2)]\cos(y-\theta_0+\omega_0) \\
+\; & \frac{1}{24}[-(14+5e_0^2)s^2 + 2e_0^2]\cos(y-3\theta_0+\omega_0) \\
+\; & \frac{e_0^2}{48}(9s^2-4)\cos(y+3\theta_0-\omega_0) + \frac{e_0^2}{8}(-7s^2+6)\cos(y+\theta_0-\omega_0) \\
+\; & \frac{e_0^2}{16}(-5s^2+4)\cos(y-\theta_0-\omega_0) + \frac{e_0}{4}(2s^2-1)\cos(y+2\theta_0) \\
+\; & \frac{e_0}{4}(-3s^2+1)\cos(y-2\theta_0) + \frac{e_0}{4}(-3s^2+2)\cos y \\
+\; & e_0 s^2\cos(\theta_0+\omega_0) + \frac{e_0 s^2}{3}\cos(3\theta_0-\omega_0) \\
+\; & s^2\cos 2\theta_0\Bigg] + O(J^2, J^3\theta, \ldots)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
y &= \theta - \omega_0 + J\Bigg\{\left(\frac{5s^2}{2} - 2\right)(\theta - \theta_0) \\
&+ \frac{e_0^2(-75s^6 + 260s^4 - 296s^2 + 112)\sin\left[J\theta\left(\frac{5s^2}{2} - 2\right)\right]\cos\left[2\omega_0 - J\theta\left(\frac{5s^2}{2} - 2\right)\right]}{24(5s^2 - 4)^2} \\
&+ J^2\theta\Bigg\{\frac{e_0^2 s^2(-15s^2 + 14)(15s^2 - 13)\cos 2\omega_0}{24(5s^2 - 4)} + \frac{e_0 s^2}{2}(15s^2 - 13)\cos(\theta_0 + \omega_0) \\
&+ \frac{e_0 s^2}{6}(15s^2 - 13)\cos(3\theta_0 - \omega_0) + \frac{s^2}{2}(15s^2 - 13)\cos 2\theta_0 \qquad (41) \\
&+ \frac{1}{96}[5(9e_0^2 + 34)s^4 + 4(9e_0^2 - 34)s^2 - 56e_0^2]\Bigg\} + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
i &= i_0 + scJ\Bigg\{\frac{1}{2}\cos 2\theta + \frac{e_0}{6}\cos(y + 2\theta) + \frac{e_0}{2}\cos(y - 2\theta) \\
&+ \frac{e_0^2(-15s^2 + 14)\sin\left[J\theta\left(\frac{5s^2}{2} - 2\right)\right]\sin\left[2\omega_0 - J\theta\left(\frac{5s^2}{2} - 2\right)\right]}{12(5s^2 - 4)} \qquad (42) \\
&- \frac{1}{2}\cos 2\theta_0 - \frac{e_0}{6}\cos(3\theta_0 - \omega_0) - \frac{e_0}{2}\cos(\theta_0 + \omega_0)\Bigg\} + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= \Omega_0 + cJ\Bigg\{\theta_0 - \theta + \frac{1}{2}\sin 2\theta - e_0\sin y \\
&+ \frac{e_0}{6}\sin(y + 2\theta) - \frac{e_0}{2}\sin(y - 2\theta) - \frac{1}{2}\sin 2\theta_0 \\
&+ e_0\sin(\theta_0 - \omega_0) - \frac{e_0}{6}\sin(3\theta_0 - \omega_0) - \frac{e_0}{2}\sin(\theta_0 + \omega_0) \\
&+ \frac{e_0^2(15s^4 - 45s^2 + 28)\sin\left[J\theta\left(\frac{5s^2}{2} - 2\right)\right]\cos\left[2\omega_0 - J\theta\left(\frac{5s^2}{2} - 2\right)\right]}{6(5s^2 - 4)^2} \\
&+ cJ^2\theta\Bigg\{\frac{e_0^2 s^2(15s^2 - 14)}{12(5s^2 - 4)}\cos 2\omega_0 - e_0 s^2\cos(\theta_0 + \omega_0) \qquad (43) \\
&- \frac{e_0 s^2}{3}\cos(3\theta_0 - \omega_0) - s^2\cos 2\theta_0 + \frac{e_0^2}{24}(7s^2 - 4) \\
&+ \frac{1}{12}(-s^2 + 6)\Bigg\} + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
t &= t_0 + \frac{1}{h_0}\int_{\theta_0}^{\theta} r^2 \Big\{ 1 + J\Big[\frac{(-3s^2+2)}{2}\cos 2\theta \\
&+ e_0(s^2-1)\cos y + \frac{e_0(-4s^2+3)}{6}\cos(y+2\theta) \\
&+ \frac{e_0(-2s^2+1)}{2}\cos(y-2\theta) \\
&+ \frac{e_0^2 s^2(15s^2-14)\sin\left[J\theta\left(\frac{5s^2}{2}-2\right)\right]\sin\left[2\omega_0 - J\theta\left(\frac{5s^2}{2}-2\right)\right]}{12(5s^2-4)} \\
&+ s^2 - 1 + \frac{s^2}{2}\cos 2\theta_0 + \frac{e_0 s^2}{6}\cos(3\theta_0-\omega_0) + \frac{e_0 s^2}{2}\cos(\theta_0+\omega_0)\Big] \\
&+ O(J^2, J^3\theta, \ldots)\Big\} d\theta
\end{aligned} \tag{44}
$$

To check the solution, we can see if the specific mechanical energy (18) of the satellite remains constant. Substitution of the solution (40)-(42) into Equations (10) and (11) yields

$$
T + V = -\frac{GM(1-\epsilon_0^2)}{2p_0} - \frac{GMJ_2R^2(1-3\sin^2\beta_0)}{2[r(t_0)]^3} + O(J^2, J^3, \ldots)
$$

The first two terms on the right side are easily recognized as the value of the specific mechanical energy at the initial time t_0.

As a further check on the solution, we can see if it reduces to previous results for equatorial and polar orbits, obtained by Danielson and Snider [Ref. 8]. Setting $i_0 = 0$ and using the independent variable α measured from the line $O\gamma$, we find that Equations (40)-(42) reduce to Equations (18)-(22) in [Ref. 8]. Setting $i_0 = \pi/2$ and using the expansion $\cos(y + Jk) \approx \cos y - Jk\sin y$, we find that Equations (40)-(42) reduce to Equations (38)-(41) in [Ref. 8].[1]

[1] There are misprints in Equations (37)-(40) of [8]. The term $2\epsilon_0 c_2 \cos y$ should be added on the right of Equation (37). The third from last term in Equation (38) should be $e_0/3\cos(3\beta_0 - \omega_0)$. The term $-c_2 J^2\beta$ should be added on the right of Equation (39). The sign of the second trigonometric term in the expression for c_6 should be changed: $(-\epsilon_0/12 + \epsilon_0^3/32)\sin(\beta_0 + \omega_0)$.

F. THE CRITICAL INCLINATION

There exist many "solutions" to the main problem. Most of these are unsatisfactory. Some are so abstruse as to have little or no physical meaning, while others suffer from having a restricted domain in inclination, eccentricity, or both. Chief among the difficulties with proposed solutions is their behavior at certain inclinations. The most persistent of these troublesome inclinations is the so-called *critical* inclination $i_{\text{crit}} \equiv \pm \arcsin 2/\sqrt{5} \pmod{\pi}$. This manifests itself in the form of denominator terms like $(4 - 5\sin^2 i)$, $(1 - 5\cos^2 i)$, or $(\tan i - 2)$. When i assumes this critical inclination, i_{crit}, each of these terms become zero, rendering the solutions useless. It is at this point that we must confront the divisor $(5s^2 - 4)$ in Equations (40)-(44). As remarked, each of (40)-(44) has a finite limit at the critical inclination. How does this come about? At the critical inclination, the quotient $\sin(J\theta(5s^{2/2} - 2)/(5s^2 - 4)$ is replaced by the limit, yielding a term in $J\theta$. This, in the parlance of orbital mechanics, is an odious *secular* term — an unbounded term which grows with the time-like variable. The expressions (40)-(44) at the critical inclination are:

$$
\begin{aligned}
r &= p_0/\Bigg\{1 + \epsilon_0 \cos y + J\bigg[1 - \frac{3s^2}{2} + \epsilon_0^2\left(1 - \frac{5s^2}{4}\right) \\
&+ +\frac{1}{12}(-(2 + 5\epsilon_0^2)s^2 + 2\epsilon_0^2)\cos 2\theta + \frac{\epsilon_0^2}{12}(9s^2 - 8)\cos 2y \\
&+ \frac{\epsilon_0^2}{24}(-11s^2 + 6)\cos(y + 2\theta) + \frac{\epsilon_0^2}{24}(-3s^2 + 2)\cos(2y + 2\theta) \\
&+ \frac{\epsilon_0^2}{8}(3s^2 - 2)\cos(2y - 2\theta) \\
&- \frac{\epsilon_0^2 s^2}{16}\cos(y - \theta_0 + 3\omega_0) + \frac{\epsilon_0^2}{24}(3s^2 - 2)\cos(y - 3\theta_0 + 3\omega_0) \\
&- \frac{\epsilon_0^2 s^2}{16}\cos(y - 5\theta_0 + 3\omega_0) + \frac{\epsilon_0}{4}(3s^2 - 2)\cos(y - 2\theta_0 + 2\omega_0) \\
&- \frac{3\epsilon_0 s^2}{8}\cos(y - 4\theta_0 + 2\omega_0) - \frac{\epsilon_0}{4}(s^2 + 1)\cos(y + 2\omega_0)
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&+ \frac{1}{8}[(-2+5\epsilon_0^2)s^2 - 2e_0^2]\cos(y+\theta_0+\omega_0) \\
&+ \frac{1}{4}[(6+5\epsilon_0^2)s^2 - 4(1+e_0^2)]\cos(y-\theta_0+\omega_0) \\
&+ \frac{1}{24}[-(14+5\epsilon_0^2)s^2 + 2e_0^2]\cos(y-3\theta_0+\omega_0) \\
&+ \frac{e_0^2}{48}(9s^2-4)\cos(y+3\theta_0-\omega_0) + \frac{e_0^2}{8}(-7s^2+6)\cos(y+\theta_0-\omega_0) \\
&+ \frac{e_0^2}{16}(-5s^2+4)\cos(y-\theta_0-\omega_0) + \frac{e_0}{4}(2s^2-1)\cos(y+2\theta_0) \\
&+ \frac{e_0}{4}(-3s^2+1)\cos(y-2\theta_0) + \frac{e_0}{4}(-3s^2+2)\cos y \\
&+ \epsilon_0 s^2\cos(\theta_0+\omega_0) + \frac{e_0^2 s^2}{3}\cos(3\theta_0-\omega_0) + s^2\cos 2\theta_0\Big] \\
&+ J^2\theta\Big[\frac{e_0}{24}(15(2+\epsilon_0^2)s^4 - 14(4+e_0^2)s^2 + 24)\sin(\theta+\omega_0) \\
&+ \frac{e_0^2 s^2}{12}(15s^2-14)\sin(2\omega_0)\Big] + O(J^2, J^3\theta, \ldots)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
y &= \theta - \omega_0 + J\left\{\left(\frac{5s^2}{2}-2\right)(\theta-\theta_0)\right\} \\
&+ J^2\theta\Big\{\frac{e_0^2}{48}(-105s^4+130s^2-28)\cos 2\omega_0 + \frac{\epsilon_0 s^2}{2}(15s^2-13)\cos(\theta_0+\omega_0) \\
&+ \frac{e_0 s^2}{6}(15s^2-13)\cos(3\theta_0-\omega_0) + \frac{s^2}{2}(15s^2-13)\cos 2\theta_0 \\
&+ \frac{1}{96}\Big[5(9\epsilon_0^2+34)s^4 + 4(9\epsilon_0^2-34)s^2 - 56\epsilon_0^2\Big]\Big\} + O(J^2, J^3\theta, \ldots)
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
i &= i_0 + scJ\Big\{\frac{1}{2}\cos 2\theta + \frac{e_0}{6}\cos(y+2\theta) + \frac{e_0}{2}\cos(y-2\theta) \\
&- \frac{1}{2}\cos 2\theta_0 - \frac{e_0}{6}\cos(3\theta_0-\omega_0) - \frac{e_0}{2}\cos(\theta_0+\omega_0)\Big\} \\
&+ J^2\theta\frac{sce_0^2}{24}(-15s^2+14)\sin 2\omega_0 + O(J^2, J^3\theta, \ldots)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\Omega &= \Omega_0 + cJ\Big\{\theta_0 - \theta + \frac{1}{2}\sin 2\theta - e_0 \sin y \\
&+ \frac{e_0}{6}\sin(y+2\theta) - \frac{e_0}{2}\sin(y-2\theta) - \frac{1}{2}\sin 2\theta_0 \\
&+ e_0\sin(\theta_0-\omega_0) - \frac{e_0}{6}\sin(3\theta-\omega_0) - \frac{e_0}{2}\sin(\theta_0-\omega_0)\Big\} \\
&+ cJ^2\theta\Big\{\frac{e_0^2}{12}(6s^2-7)\cos(2\omega_0) - e_0 s^2\cos(\theta_0+\omega_0) \qquad (48)\\
&- \frac{e_0 s^2}{3}\cos(3\theta_0-\omega_0) - s^2\cos 2\theta_0 + \frac{e_0^2}{24}(7s^2-4) \\
&+ \frac{1}{12}(-s^2+6)\Big\} + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
t &= t_0 + \frac{1}{h_0}\int_{\theta_0}^{\theta} r^2\Big\{1 + J\Big[\frac{(-3s^2+2)}{2}\cos 2\theta \\
&+ e_0(s^2-1)\cos y + \frac{e_0(-4s^2+3)}{6}\cos(y+2\theta) \\
&+ \frac{e_0(-2s^2+1)}{2}\cos(y-2\theta) \qquad (49)\\
&+ s^2 - 1 + \frac{s^2}{2}\cos 2\theta_0 + \frac{e_0 s^2}{6}\cos(3\theta_0-\omega_0) + \frac{e_0 s^2}{2}\cos(\theta_0+\omega_0)\Big] \\
&+ J^2\theta\frac{e_0^2 s^2}{24}(15s^2-14)\sin(2\omega_0) + O(J^2, J^3\theta, \ldots)\Big\}d\theta
\end{aligned}
$$

As can be seen, none of the variables fails to be defined at any inclination or eccentricity.

A long-standing debate in astrodynamics centers about the nature of the critical inclination's ubiquitous presence in proffered solutions. Of course something important *does* occur at the critical inclination — the line of apsides remains fixed. That this event should also cause most attempts at solving the problem to fail has been the subject of much debate and, apparently, misunderstanding.

Examination of Equation (45) shows that when i assumes the critical inclination there result terms in $J^2\theta$. At the order to which we have approximated the

solution the *form* of our approximation at the critical inclination differs from that of all inclinations. In fact, if $i \neq i_{\text{crit}}$, it appears that we can obtain arbitrary accuracy for sufficiently small J and θ in some J dependent interval $[0, c/J^m]$, (m constant, integer, c an $O(1)$ constant), by taking sufficiently many terms. However, when $i = i_{\text{crit}}$, the expansion has a restricted range of validity, for θ in $[0, c/J]$ where c is an $O(1)$ constant. In other words, while we have dropped the higher order terms and higher order secular terms in our solution, it is hoped that away from the critical inclination it will be possible to improve the approximation to arbitrary accuracy simply by continuing the method used. Not so at the critical inclination—we are already confronted with a secular term which cannot be eliminated, regardless of how many terms we keep in our approximation. We are thus in agreement with Coffey, Deprit, and Miller [Ref. 9] that the critical inclination is an intrinsic singularity, independent of the method used to solve the problem. Moreover, there may emerge other critical inclinations as higher order analyses are performed. It may be that a higher order approximation will yield other inclinations where secular terms arise which cannot be eliminated. The well-known critical inclination might be only the first in a sequence.

Enters now the astrodynamicist, the man in the field. When the equations of motion are integrated at the critical inclination, no strange effects are reported. The mathematicians have declared the critical inclination a glaring singularity, yet everyday practitioners consider it a trifling locale. If a satellite were to start an orbit at the critical inclination, Equation (45) predicts a secular variation in $u(\theta)$ of order $J^2\theta$. Clearly such slow growth with θ explains why no appreciable effect of the critical inclination has been encountered in the intensive numerical integrations which have been carried out over the years. Additionally, if a satellite is initially at the critical inclination, Equation (47) states that the expression for its inclination

also contains a term in $J^2\theta$. This means that the satellite will tend away from the critical before the secular terms which arise in $u(\theta)$ at the critical become important. As the satellite moves away from the critical inclination, the terms in $u(\theta)$ resume their bounded trigonometric form. From a practical viewpoint, the critical inclination problem is in fact a nonproblem. The simplifications and assumptions made will perforce limit the time over which the solution is valid, further obscuring attempts to physically observe anomalies at the critical inclination. We must concede that other perturbative forces would play a major role in determining the degree to which the critical inclination could induce observable effects. This is not to endorse a variety of patchwork measures now being used to determine satellite motion, such as "gapping" the critical inclination in order to avoid dividing by zero. The critical inclination, delicate as though it may be, is perhaps a powerful telltale for faulty modeling.

G. CONCLUSIONS

As shown, our solution satisfies the goals given in the introduction.

If the perturbation procedure is continued, it is anticipated that each of the additional terms added to Equations (40)-(44) will be multiplied by one of the factors $(J^2, J^3, J^3\theta, J^4\theta, \ldots)$. If we restrict $\theta \leq 1/J$, the neglected terms should be order J^2. (For an Earth satellite $J < 3/2 \times 10^{-3}$, so for at least 100 revolutions the relative error should be order 10^{-6}.)

If we restrict $\theta \leq 1$, all of the terms of order $J^2\theta$ in the solution (40)-(44) can be dropped without increasing the order of magnitude of the error. This considerably simplifies the solution to:

$$
\begin{aligned}
r &= p_0/\Big\{1 + e_0\cos\left[\theta - \omega_0 + J(\theta-\theta_0)\left(\frac{5s^2}{2} - 2\right)\right] \\
&+ J\Big[1 - \frac{3s^2}{2} + e_0^2\left(1 - \frac{5s^2}{4}\right) + \frac{1}{12} - (2 + 5e_0^2)s^2 + 2e_0^2)\cos 2\theta \\
&+ \frac{e_0^2}{12}(9s^2 - 8)\cos(2\theta - 2\omega_0) + \frac{e_0}{24}(-11s^2 + 6)\cos(3\theta - \omega_0) \\
&+ \frac{e_0^2}{24}(-3s^2 + 2)\cos(4\theta - 2\omega_0) + \frac{e_0^2}{8}(3s^2 - 2)\cos 2\omega_0 \\
&- \frac{e_0^2 s^2}{16}\cos(\theta - \theta_0 + 2\omega_0) + \frac{e_0^2}{24}(3s^2 - 2)\cos(\theta - 3\theta_0 + 2\omega_0) \\
&- \frac{e_0^2 s^2}{16}\cos(\theta - 5\theta_0 + 2\omega_0) + \frac{e_0}{4}(3s^2 - 2)\cos(\theta - 2\theta_0 + \omega_0) \\
&- \frac{3e_0 s^2}{8}\cos(\theta - 4\theta_0 + \omega_0) - \frac{e_0}{4}(s^2 + 1)\cos(\theta + \omega_0) \\
&+ \frac{1}{8}[(-2 + 5e_0^2)s^2 - 2e_0^2]\cos(\theta + \theta_0) \\
&+ \frac{1}{4}[(6 + 5e_0^2)s^2 - 4(1 + e_0^2)]\cos(\theta - \theta_0) \\
&+ \frac{1}{24}[-(14 + 5e_0^2)s^2 + 2e_0^2]\cos(\theta - 3\theta_0) \\
&+ \frac{e_0^2}{48}(9s^2 - 4)\cos(\theta + 3\theta_0 - 2\omega_0) + \frac{e_0^2}{8}(-7s^2 + 6)\cos(\theta + \theta_0 - 2\omega_0) \\
&+ \frac{e_0^2}{16}(-5s^2 + 4)\cos(\theta - \theta_0 - 2\omega_0) + \frac{e_0}{4}(2s^2 - 1)\cos(\theta + 2\theta_0 - \omega_0) \\
&+ \frac{e_0}{4}(-3s^2 + 1)\cos(\theta - 2\theta_0 - \omega_0) \\
&+ \frac{e_0}{4}(-3s^2 + 2)\cos(\theta - \omega_0) + e_0 s^2\cos(\theta_0 + \omega_0) \\
&+ \frac{e_0 s^2}{3}\cos(3\theta_0 - \omega_0) + s^2\cos 2\theta_0\Big] + O(J^2, J^3\theta, \ldots)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
i &= i_0 + scJ\Big[\frac{1}{2}\cos 2\theta + \frac{e_0}{6}\cos(3\theta - \omega_0) + \frac{e_0}{2}\cos(\theta + \omega_0) \\
&- \frac{1}{2}\cos 2\theta_0 - \frac{e_0}{6}\cos(3\theta_0 - \omega_0) - \frac{e_0}{2}\cos(\theta_0 + \omega_0)\Big] + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= \Omega_0 + cJ\Big[\theta_0 - \theta + \frac{1}{2}\sin 2\theta - e_0\sin(\theta-\omega_0) + \frac{e_0}{6}\sin(3\theta-\omega_0) \\
&+ \frac{e_0}{2}\sin(\theta+\omega_0) - \frac{1}{2}\sin 2\theta_0 + e_0\sin(\theta_0-\omega_0) - \frac{e_0}{6}\sin(3\theta_0-\omega_0) \\
&- \frac{e_0}{2}\sin(\theta_0+\omega_0)\Big] + O(J^2, J^3\theta, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
t &= t_0 + \frac{1}{h_0}\int_{\theta_0}^{\theta} r^2\Big\{1 + J\Big[\frac{(-3s^2+2)}{2}\cos 2\theta \\
&+ e_0(s^2-1)\cos(\theta-\omega_0) + \frac{e_0(-4s^2+3)}{6}\cos(3\theta-\omega_0) \\
&+ \frac{e_0(-2s^2+1)}{2}\cos(\theta+\omega_0) + s^2 - 1 + \frac{s^2}{2}\cos 2\theta_0 \\
&+ \frac{e_0s^2}{6}\cos(3\theta_0-\omega_0) + \frac{e_0s^2}{2}\cos(\theta_0+\omega_0)\Big] \\
&+ O(J^2, J^3\theta, \ldots)\Big\}d\theta
\end{aligned}
$$

APPENDIX

$$K_1 = \frac{cse_0^2(-15s^2+14)}{24(5s^2-4)}$$

$$\begin{aligned} K_2 &= -\frac{sc}{2}\cos 2\theta_0 - \frac{sce_0}{6}\cos(3\theta_0-\omega_0) - \frac{sce_0}{2}\cos(\theta_0+\omega_0) \\ &+ \frac{cse_0^2(15s^2-14)}{24(5s^2-4)}\cos 2\omega_0 \end{aligned}$$

$$K_3 = \frac{e_0^2(-75s^6+260s^4-296s^2+112)}{48(5s^2-4)^2}$$

$$K_4 = e_0\frac{[15(e_0^2+2)s^4-14(e_0^2+4)s^2+24]}{5s^2-4}$$

$$\begin{aligned} K_5 &= \frac{e_0^2}{12}(-9s^2+8)\cos(2\theta_0-2\omega_0) + \frac{e_0^2}{24}(3s^2-2)\cos(4\theta_0-2\omega_0) \\ &- (e_0s^2+K_4)\cos(\theta_0+\omega_0) + \frac{e_0}{8}(s^2-2)\cos(3\theta_0-\omega_0) \\ &+ \frac{e_0^2}{8}(-3s^2+2)\cos 2\omega_0 - \frac{1}{12}[5(2-e_0^2)s^2+2e_0^2]\cos 2\theta_0 \\ &+ \frac{1}{12}(15e_0^2+18)s^2 - (e_0^2+1) \end{aligned}$$

$$\begin{aligned} K_6 &= +\frac{e_0^2}{6}(6s^2-5)\sin(2\theta_0-2\omega_0) + \frac{e_0^2}{12}(-3s^2+1)\sin(4\theta_0-2\omega_0) \\ &+ \frac{1}{2}[e_0(-s^2+1)+2K_4]\sin(\theta_0+\omega_0) + \frac{e_0}{2}(3s^2-2)\sin(\theta_0-\omega_0) \\ &+ \frac{e_0}{8}(-7s^2+2)\sin(3\theta_0-\omega_0) + \frac{e_0^2}{4}(-s^2+1)\sin 2\omega_0 \\ &+ \frac{1}{6}[-(5e_0^2+2)s^2+2e_0^2] \end{aligned}$$

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